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Variational and rigidity properties of static potentials

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In this paper we study some global properties of static potentials on asymptotically flat 3-manifolds (M, g) in the nonvacuum setting. Heuristically, a static potential f represents the (signed) length along M of an irrotational timelike Killing vector field, which can degenerate on surfaces corresponding to the zero set of f. Assuming a suitable version of the null energy condition, we prove that a noncompact component of the zero set must be area minimizing. From this we obtain some rigidity results for static potentials that have noncompact zero set components, or equivalently, that are unbounded. Roughly speaking, these results show, at the pure initial data level, that 'boost-type' Killing vector fields can exist only under special circumstances.

1. Introduction

Consider a static spacetime

$$\bar{M} = \mathbb{R} \times M\,, \quad \bar{g} = -f^2 dt^2 + g, \tag{1.1}$$

where (M, g) is a Riemannian 3-manifold and f is a positive function on M. The function f and the Ricci tensors $\mathrm{Ric}_{\bar{M}}$, and Ric, of $(\bar{M}, \bar{g})$, and (M, g), respectively, are related by,

$$\nabla^2 f = f(\mathrm{Ric} - \mathrm{Ric}_{\bar{M}}|_M)\,, \tag{1.2}$$

$$\triangle f = \mathrm{Ric}_{\bar{M}}(u, u) f\,, \tag{1.3}$$

where $\mathrm{Ric}_{\bar{M}}|_M$ denotes the restriction of $\mathrm{Ric}_{\bar{M}}$ to the tangent space of M and $u = f^{-1}\partial_t$ is the future timelike unit normal to M in $(\bar{M}, \bar{g})$. When $(\bar{M}, \bar{g})$ satisfies the Einstein equation, then Equations (1.2) and (1.3) are the (in general nonvacuum) Einstein field equations for a static spacetime.

Let $R_{\bar{M}}$ and R be the scalar curvature of $(\bar{M}, \bar{g})$ and (M, g), respectively. In terms of the Einstein tensor,

$$G = \mathrm{Ric}_{\bar{M}} - \frac{1}{2} R_{\bar{M}}\, \bar{g}\,, \tag{1.4}$$

Equations (1.2) and (1.3) become,

$$\nabla^2 f = f\left[\mathrm{Ric} - \gamma + \frac{1}{2}(\mathrm{tr}\gamma - \rho) g\right], \tag{1.5}$$

$$\triangle f = \frac{1}{2}(\rho + \mathrm{tr}\gamma) f\,, \tag{1.6}$$

where $\rho = G(u, u) = \frac{1}{2}R$, and γ is G restricted to TM. If one assumes the Einstein equation holds: $G = T$, where T is the energy-momentum tensor, then decay conditions on ρ and γ may be viewed as decay conditions on T.

More generally, for a given Riemannian 3-manifold (M, g), scalar field ρ and symmetric 2-tensor γ, a nontrivial smooth function f, without any sign assumptions, will be called a *static potential* for (M, g, ρ, γ) provided Equations (1.5) and (1.6) hold. Static potentials in the *vacuum* case ($\rho = 0$, $\gamma = 0$) arose in the work of Corvino [10], where they correspond to nontrivial elements in the kernel of the adjoint of the linearized scalar curvature map (see also [11]). From a slightly different point of view, in studying static potentials, we are in essence considering Killing initial data [4] with zero shift.

In this paper we establish some global properties of static potentials for asymptotically flat 3-manifolds (M, g), subject to natural energy and decay conditions on ρ and γ. In Section 2 we present some local and asymptotic properties of static potentials. In Section 3 we establish a basic restriction on the occurrence of minimal surfaces having boundaries which lie on the zero set of a static potential; cf. Theorem 3.1. This result is used in Section 4 to show that a noncompact component of the zero set must be area minimizing; cf. Theorem 4.1. From this we obtain some nonexistence and rigidity results for static potentials that have noncompact zero set components, or equivalently, that are unbounded. Roughly speaking, these results show, at the pure initial data level, that ‘boost-type’ Killing vector fields can exist only under special circumstances. See [5, Theorem 1.1] for a related spacetime result, which applies to asymptotically flat spacetimes that admit boost domains.

2. Preliminaries

In this section, we collect some preliminary results concerning a nontrivial solution f to

$$\nabla^2 f = f\left[\mathrm{Ric} - \gamma + \frac{1}{2}(\mathrm{tr}\gamma - \rho)g\right] \tag{2.1}$$

on (M, g, γ, ρ), where M is always assumed to be connected.

We start with some local properties of the zero set of f. The following lemma is an analogue of [18, Lemma 2.1] (which focused on the case $\rho = 0$ and $\gamma = 0$).

Lemma 2.1. *Suppose f is a nontrivial solution to* (2.1). *Let $\Sigma = f^{-1}(0)$. Suppose $\Sigma \neq \emptyset$.*

(i) *Σ is a totally geodesic hypersurface and $|\nabla f|$ is a positive constant on each connected component of Σ.*

(ii) *∇f is an eigenvector of Ric along Σ.*

(iii) *Suppose $\rho = 0$ and $\gamma = 0$ along Σ. At any $p \in \Sigma$, let $\{e_1, e_2, e_3\}$ be an orthonormal frame such that e_3 is normal to Σ. Let R_{ijkl} denote the component of the curvature tensor in this frame such that R_{ijij} equals the sectional curvature of the tangent 2-plane spanned by e_i and e_j for $i \neq j \in \{1, 2, 3\}$. Then*

$$R_{1313} = R_{2323} = -\frac{1}{2}R_{1212}.$$

As a result, $R = 0$ along Σ where R is the scalar curvature (M, g) and

$$Ric|_{T\Sigma} = \frac{1}{2}Kg|_{T\Sigma}, \quad Ric(e_3, e_3) = -K, \tag{2.2}$$

where $Ric|_{T\Sigma}$, $g|_{T\Sigma}$ denote the restriction of Ric, g to the tangent space to Σ respectively, and K is the the Gaussian curvature of Σ.

Proof. (i) Let $p \in \Sigma$. If $\nabla f(p) = 0$, then along any geodesic $\beta(t)$ emanating from p, $f(t) := f(\beta(t))$ satisfies

$$f''(t) = h(t)f(t), \quad f'(0) = 0, \quad f(0) = 0,$$

where $h(t) = \mathrm{Ric}(\beta'(t), \beta'(t)) - \gamma(\beta'(t), \beta'(t)) + \frac{1}{2}(\mathrm{tr}\gamma - \rho)(\beta(t))$. This implies f is zero near p. Now consider the set

$$\Lambda = \{x \in M \mid f(x) = 0 \text{ and } \nabla f(x) = 0\}.$$

Clearly, Λ is a closed set. The above argument shows that Λ is also open. Since $p \in \Lambda$ and M is connected, we conclude $\Lambda = M$, contradicting the fact that f is nontrivial. Therefore $\nabla f(p) \neq 0$, $\forall\ p \in \Sigma$, which implies Σ is an embedded surface. Along Σ, (1.5) shows $\nabla^2 f(X, Y) = 0$ and $\nabla^2 f(X, \nabla f) = 0$ for any tangent vectors X, Y tangential to Σ. This readily implies that Σ is totally geodesic and $|\nabla f|^2$ is a constant along Σ.

(ii) Since Σ is totally geodesic, it follows from the Codazzi equation that

$$\mathrm{Ric}(\nu, X) = 0 \tag{2.3}$$

for all X tangent to Σ, where ν is the unit normal of Σ. Therefore, $\nabla f = \frac{\partial f}{\partial \nu}\nu$ is an eigenvector of Ric.

(iii) Let $R_{ij} = \mathrm{Ric}(e_i, e_j)$, where $i, j \in \{1, 2, 3\}$. Differentiating (2.1) and applying the fact $\rho = 0$ and $\gamma = 0$ along Σ, we have $f_{;ijk} = f_{;k}R_{ij}$. Setting $k = 3$ gives

$$f_{;ij3} = f_{;3}R_{ij}.$$

On the other hand,

$$f_{;ij3} - f_{;i3j} = -\sum_{l=1}^{3} R_{li3j} f_{;l} = -R_{3i3j} f_{;3}. \tag{2.4}$$

Therefore,

$$f_{;3}R_{\alpha\beta} = f_{;\alpha 3\beta} - R_{3\alpha 3\beta} f_{;3} = -R_{3\alpha 3\beta} f_{;3}, \tag{2.5}$$

where $\alpha, \beta \in \{1, 2\}$ and we used the fact $f_{;ij} = 0$ along Σ. Since $f_{;3} \neq 0$, (2.5) implies

$$R_{\alpha\beta} = -R_{3\alpha 3\beta}. \tag{2.6}$$

By the definition of $R_{\alpha\beta}$, this is equivalent to

$$R_{1\alpha 1\beta} + R_{2\alpha 2\beta} + R_{3\alpha 3\beta} = -R_{3\alpha 3\beta}. \tag{2.7}$$

By taking $\alpha = \beta = 1$ and 2 respectively, (2.7) implies

$$R_{3131} = R_{3232} = -\frac{1}{2}R_{1212}, \tag{2.8}$$

where $R_{1212} = K$ since Σ is totally geodesic. This completes the proof. $\square$

Next, we assume that M is diffeomorphic to $\mathbb{R}^3 \setminus B$, where B is an open Euclidean ball centered at the origin, and g is a smooth metric on M such that with respect to the standard coordinates $\{x_i\}$ on $\mathbb{R}^3$, g satisfies

$$g_{ij} = \delta_{ij} + b_{ij} \quad \text{with} \quad b_{ij} = O_2(|x|^{-\tau}) \tag{2.9}$$

for some constant $\tau \in (\frac{1}{2}, 1]$. We also assume the symmetric $(0,2)$ tensor γ and the scalar field ρ satisfy

$$\gamma_{ij} = O(|x|^{-2-\tau}), \quad \rho = O(|x|^{-2-\tau}). \tag{2.10}$$

The following proposition concerning the asymptotic behavior of f near infinity follows from [3, Proposition 2.1] (by setting the shift vector Y in [3] equal to zero).

Proposition 2.1. *Suppose f is a solution to* (2.1) *on* (M, g, ρ, γ) *satisfying* (2.9) *and* (2.10). *Then*

(i) *there exists a tuple* $(a_1, a_2, a_3) \in \mathbb{R}^3$ *such that*

$$f = a_1 x_1 + a_2 x_2 + a_3 x_3 + h$$

where h satisfies $\partial h = O_1(|x|^{-\tau})$ *and*

$$|h| = \begin{cases} O(|x|^{1-\tau}) & \text{when } \tau < 1, \\ O(\ln |x|) & \text{when } \tau = 1. \end{cases}$$

(ii) $(a_1, a_2, a_3) = (0, 0, 0)$ *if and only if f has a finite limit at ∞. In this case,* $\lim_{x\to\infty} f = 0$ *only if f is identically zero.*

Proposition 2.1 itself implies that the zero set of an unbounded f near infinity has a controlled graphical structure.

Proposition 2.2. *Suppose f is an unbounded solution to* (2.1) *on* (M,g,ρ,γ) *satisfying* (2.9) *and* (2.10). *Then there exists a new set of coordinates $\{y_i\}$ near the infinity of (M, g), obtained by a rotation of $\{x_i\}$ such that, outside a compact set, $f^{-1}(0)$ is given by the graph of a smooth function $q = q(\bar{y})$,* *where* $\bar{y} = (y_2, y_3)$*, over*

$$\Omega_C = \{(0, y_2, y_3) \mid y_2^2 + y_3^2 > C^2\}$$

for some constant $C > 0$*, where* q *satisfies*

$$\partial q = O_1(|\bar{y}|^{-\tau}) \quad \text{and} \quad |q| = \begin{cases} O(|\bar{y}|^{1-\tau}) & \text{when } \tau < 1 \\ O(\ln |\bar{y}|) & \text{when } \tau = 1. \end{cases} \tag{2.11}$$

As a result, if $\gamma_{_R} \subset f^{-1}(0)$ *is the curve given by*

$$\gamma_{_R} = \{(q(y_2, y_3), y_2, y_3) \mid y_2^2 + y_3^2 = R^2\}$$

and κ *is the geodesic curvature of* $\gamma_{_R}$ *in* $f^{-1}(0)$*, then*

$$\lim_{R\to\infty} \int_{\gamma_{_R}} \kappa \; ds = 2\pi. \tag{2.12}$$

Proposition 2.2 follows from Proposition 2.1 by the exact same proof in [18] that proves [18, Proposition 3.2] using [18, Proposition 3.1].

3. Static potentials and minimal surfaces

In this section, we assume (M, g) is a complete, asymptotically flat 3-manifold without boundary, with finitely many ends. The triple (g, γ, ρ) is assumed to satisfy the decay assumptions (2.9) and (2.10) on each end.

We say that (M, g, γ, ρ) satisfies the null energy condition (NEC) provided,

$$\rho + \gamma(X, X) \geq 0 \quad \text{for all unit vectors } X. \tag{3.1}$$

For the static metric (1.1), this is equivalent to requiring $\mathrm{Ric}_{\bar{M}}(K, K) \geq 0$ for all null vectors K along M.

The aim of this section is to establish the following theorem.

Theorem 3.1. *Assume* (M, g, γ, ρ) *satisfies the NEC and* f *is a static potential. Let* U *be an unbounded, connected component of* $\{f \neq 0\}$*. Then there does not exist any compact subset* $S \subset M$ *such that* $S \setminus \partial U$ *is an embedded minimal surface in* U*.*

In other words, under the given assumptions, a compact minimal surface whose boundary lies on the zero set of f cannot penetrate an unbounded component of $\{f \neq 0\}$.[1]

The proof makes use of two lemmas which we present first. Suppose f is a static potential on (M, g, γ, ρ). By Proposition 2.1 (i), $\lim_{x\to\infty} |\nabla f|_g$ exists and is finite at each end. Therefore,

$$\Lambda < \infty, \quad \text{where } \Lambda = \sup_M |\nabla f|_g. \tag{3.2}$$

In the following lemmas, we consider properties of the conformally deformed *Fermat* metric

$$\tilde{g} = f^{-2} g$$

on the open set $\{f \neq 0\}$.

Lemma 3.1. *Let U be a connected component of $\{f \neq 0\}$. Let $p, q \in U$ be two distinct points. Let $dist_g(p, q)$ be the distance between p and q in (M, g). Given any curve β in U that connects p and q, let $l(\beta, \tilde{g})$ denote the $\tilde{g}$-length of β. Then*

$$l(\beta, \tilde{g}) \geq \frac{1}{\Lambda} \ln \left(1 + \frac{\Lambda \, dist_g(p, q)}{\min\{|f(p)|, |f(q)|\}} \right). \tag{3.3}$$

Proof. Without loss of generality, we may assume $f > 0$ on U. Let l be the g-length of β. We parametrize β such that

$$\beta(0) = p, \quad \beta(l) = q, \quad |\beta'(t)|_g = 1, \ \forall\, t \in [0, l]. \tag{3.4}$$

The $\tilde{g}$-length of β is then given by

$$\begin{aligned} l(\beta, \tilde{g}) &= \int_0^l \frac{1}{f(\beta(t))} dt \\ &\geq \int_0^d \frac{1}{f(\beta(t))} dt, \end{aligned} \tag{3.5}$$

where $d = \mathrm{dist}_g(p, q) > 0$. By (3.4), we have

$$\left| \frac{d}{dt} f(\beta(t)) \right| = \left| \langle \nabla f, \beta'(t) \rangle_g \right| \leq \Lambda. \tag{3.6}$$

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[1]Some related results ruling out penetrating marginally outer trapped surfaces, which are closed (compact without boundary) and bounding, in static and stationary spacetimes, are obtained in [9] by different methods.

Therefore, $\forall\, t \in [0, l]$,

$$f(\beta(t)) - f(\beta(0)) \leq \Lambda t. \tag{3.7}$$

This, together with the fact $f(\beta(t)) > 0$, shows

$$\frac{1}{f(\beta(t))} \geq \frac{1}{f(\beta(0)) + \Lambda t}. \tag{3.8}$$

Hence, by (3.5),

$$\begin{aligned} l(\beta, \tilde{g}) &\geq \int_0^d \frac{1}{f(\beta(0)) + \Lambda t} dt \\ &= \frac{1}{\Lambda} \left[\ln\left(f(p) + \Lambda d\right) - \ln f(p)\right)] \\ &= \frac{1}{\Lambda} \ln\left(1 + \frac{\Lambda d}{f(p)}\right), \end{aligned} \tag{3.9}$$

Reversing the direction of β, we have

$$l(\beta, \tilde{g}) \geq \frac{1}{\Lambda} \ln\left(1 + \frac{\Lambda d}{f(q)}\right). \tag{3.10}$$

Therefore, (3.3) follows from (3.9) and (3.10). □

Lemma 3.2. *Let U be a connected component of $\{f \neq 0\}$. The metric $\tilde{g}$ is complete on U*

Proof. Let $\beta : [0, T) \to (U, \tilde{g})$ be an inextendible geodesic ray in $(U, \tilde{g})$. We want to show $T = \infty$.

Suppose $T < \infty$. Without loss of generality, we assume $|\beta'(t)|_{\tilde{g}} = 1, \forall\, t \in [0, T)$. Replacing f by $-f$, we may also assume $f > 0$ on U. Then $|\beta'(t)|_g = f(\beta(t))$ and

$$\left|\frac{d}{dt} f(\beta(t))\right| = \left|\langle \nabla f, \beta'(t)\rangle_g\right| \leq \Lambda f(\beta(t)). \tag{3.11}$$

By (3.11), we have

$$\left|\frac{d}{dt} \ln f(\beta(t))\right| \leq \Lambda, \tag{3.12}$$

which implies

$$f(\beta(t)) \leq f(\beta(0)) e^{\Lambda t}, \quad \forall\, t \in [0, T). \tag{3.13}$$

Therefore, the length l of β in (M, g) satisfies

$$l = \int_0^T f(\beta(t))dt < \infty. \tag{3.14}$$

Since (M, g) is complete, (3.14) implies

$$\lim_{t \to T-} \beta(t) = q$$

for some point $q \in \overline{U} \subset M$.

If $q \in U$, the geodesic $\beta : [0, T) \to (U, \tilde{g})$ can be extended beyond T, contradicting the inextendibility of β.

If $q \in \partial U$, then $f(q) = 0$. By Lemma 3.1, this implies the length of β in $(U, \tilde{g})$ is ∞, which contradicts the assumption $T < \infty$.

Therefore, we must have $T = \infty$, which shows $\tilde{g}$ is complete on U. □

A proof of completeness of the Fermat metric under somewhat different circumstances has been considered in [16, Lemma 3].

Proof of Theorem 3.1. Without losing generality, we assume $f > 0$ in U. Let $E_1, \ldots, E_k$ be all the ends of (M, g). For each large r, let $S_r^{(i)}$ be the coordinate sphere $\{|x| = r\}$ near infinity on the end E_i. Define $S_r = \cup_{i=1}^k S_r^{(i)}$ and $S_{r,U} = S_r \cap U$. Since U is unbounded, $S_{r,U} \neq \emptyset$.

Given any compact subset S of M such that $S \setminus \partial U$ is an embedded surface in U, define $S_U = S \setminus \partial U$. Since S is compact, $S \subset \Omega_r$ for sufficiently large r where Ω_r is the bounded open set enclosed by S_r in (M, g). Now consider the two disjoint surfaces S_U and $S_{r,U}$ in $(U, \tilde{g})$. Let $\mathrm{dist}_{\tilde{g}}(\cdot, \cdot)$ denote the distance functional on $(U, \tilde{g})$. We claim $\mathrm{dist}_{\tilde{g}}(S_U, S_{r,U}) > 0$ and there exists $p \in S_U$ and $q \in S_{r,U}$ such that $\mathrm{dist}_{\tilde{g}}(p, q) = \mathrm{dist}_{\tilde{g}}(S_U, S_{r,U})$.

To prove the above claim, suppose $\{p_k\} \subset S_U$, $\{q_k\} \subset S_{r,U}$ are sequences of points such that

$$\lim_{k \to \infty} \mathrm{dist}_{\tilde{g}}(p_k, q_k) = \mathrm{dist}_{\tilde{g}}(S_U, S_{r,U}). \tag{3.15}$$

By Lemma 3.1, we have

$$\mathrm{dist}_{\tilde{g}}(p_k, q_k) \geq \frac{1}{\Lambda} \ln \left(1 + \frac{\Lambda \mathrm{dist}_g(S, S_r)}{\min\{|f(p_k)|, |f(q_k)|\}} \right), \tag{3.16}$$

where $\mathrm{dist}_g(S, S_r) > 0$ is the distance between S and S_r in (M, g). Since S and S_r are compact, there exists $p \in S$ and $q \in S_r$ such that, passing to a subsequence, $\lim_{k \to \infty} p_k = p$ and $\lim_{k \to \infty} q_k = q$. If either $f(p) = 0$ or

$f(q) = 0$, then (3.16) implies $\lim_{k\to\infty} \mathrm{dist}_{\tilde{g}}(p_k, q_k) = \infty$, contradicting (3.15) and the fact $\mathrm{dist}_{\tilde{g}}(S_U, S_{r,U}) < \infty$. Therefore, we must have $p \in S_U$ and $q \in S_{r,U}$. Hence,

$$\mathrm{dist}_{\tilde{g}}(p, q) = \mathrm{dist}_{\tilde{g}}(S_U, S_{r,U}). \tag{3.17}$$

Since $S_U \cap S_{r,U} = \emptyset$, we also have $\mathrm{dist}_{\tilde{g}}(S_U, S_{r,U}) > 0$.

To proceed, we apply Lemma 3.2 to conclude that there exists a unit speed $\tilde{g}$-geodesic $\beta : [0, L] \to (U, \tilde{g})$ such that

$$\beta(0) = p, \quad \beta(L) = q \quad \text{and} \quad L = \mathrm{dist}_{\tilde{g}}(S_U, S_{r,U}). \tag{3.18}$$

Since β minimizes the $\tilde{g}$-distance between points on S_U and $S_{r,U}$, there are no $\tilde{g}$-cut points to S_U along β, except possibly at the end point $q = \beta(L)$. Moreover, by the fact $S_U \subset \Omega_r$, we have $\beta([0, L)) \subset \Omega_r$. Hence, $\tilde{\mu} := \beta'(L)$ is $\tilde{g}$-normal to S_r and is outward pointing (with respect to Ω_r). As a result, $\mu := \frac{1}{f(q)}\tilde{\mu}$ is the outward unit normal vector to S_r at q in (M, g). Therefore, by the fact (M, g) is asymptotically flat, we have

$$H(S_r, q) > 0 \tag{3.19}$$

for large r, where $H(S_r, q)$ is the mean curvature of S_r with respect to μ in (M, g). Our sign convention on the mean curvature is that a Euclidean ball has positive mean curvature with respect to its outward normal.)

Next, suppose $S \setminus \partial U$ is a minimal surface in (M, g). We will derive a contradiction to (3.19) using Equations (1.5), (1.6) and the maximum principle. To illustrate the main idea used, we first consider the case that $q = \beta(L)$ is not a $\tilde{g}$-cut point to S_U along β. In this case, on the surface S_U, there exists a small open neighborhood W of p such that the map

$$\Phi(t, x) := \widetilde{\exp}_x(t\tilde{\nu}),$$

where $\widetilde{\exp}_{(\cdot)}(\cdot)$ is the $\tilde{g}$-exponential map, $t \in [0, L]$, $x \in W$ and $\tilde{\nu}$ is a $\tilde{g}$-unit vector field normal to W with $\tilde{\nu}(p) = \beta'(0)$, is a diffeomorphism from $[0, L] \times W$ onto its image in U. For each $t \in [0, L]$, let $W_t = \Phi(t, W)$ and let $H = H(t)$ be the mean curvature of W_t with respect to $\nu(t) = f^{-1}\Phi_*(\frac{\partial}{\partial t})$ in (M, g). In what follows, we perform all the computations with respect to the original metric g. (The metric $\tilde{g}$ was only used to produce the variation $t \mapsto W_t$ in U.)

We now obtain a monotonicity formula involving $H = H(t)$ (along the lines of that considered in [13]; see also [15, 16]). By a standard computation, we have

$$\frac{\partial H}{\partial t} = -\triangle_t f - (\mathrm{Ric}(\nu,\nu) + |\mathrm{I\!I}|^2)f\,, \tag{3.20}$$

where $\triangle_t$ is the Laplacian on W_t and $\mathrm{I\!I} = \mathrm{I\!I}(t)$ is the second fundamental form of W_t. The Laplacians $\triangle$ on M and $\triangle_t$ on W_t are related by

$$\triangle f = \triangle_t f + \nabla^2 f(\nu,\nu) + \frac{H}{f}\frac{\partial f}{\partial t}\,. \tag{3.21}$$

Substituting (1.5) and (1.6) into the above gives

$$\triangle_t f = [\rho + \gamma(\nu,\nu) - \mathrm{Ric}(\nu,\nu)]\, f - \frac{H}{f}\frac{\partial f}{\partial t}\,. \tag{3.22}$$

It follows from (3.20) and (3.22) that

$$\frac{\partial H}{\partial t} = -\left[|\mathrm{I\!I}|^2 + \gamma(\nu,\nu) + \rho\right] f + \frac{H}{f}\frac{\partial f}{\partial t} \tag{3.23}$$

or equivalently

$$\frac{\partial}{\partial t}\left(\frac{H}{f}\right) = -\left[|\mathrm{I\!I}|^2 + \gamma(\nu,\nu) + \rho\right]. \tag{3.24}$$

Hence by the NEC (3.1), we have

$$\frac{\partial}{\partial t}\left(\frac{H}{f}\right) \le 0\,. \tag{3.25}$$

By (3.25), if $W = W_0$ has mean curvature $H(0) \le 0$ with respect to ν, then W_L has mean curvature $H_L \le 0$. On the other hand, by the minimizing property of β, i.e. (3.18), $W_L \subset \bar{\Omega}_r$ and W_L touches S_r at q. Therefore, the inequality $H_L \le 0$ contradicts (3.19) and the maximum principle. Hence, S_U can not be a minimal surface in (M,g).

To complete the proof, we need to handle the case $q = \beta(L)$ is a $\tilde{g}$-cut point to S_U along β. We will reduce this case to the case just considered above by the following procedure. Let $\tilde{\eta}$ denote the inward, unit normal to $S_{r,U}$ in (U,g). In particular, $\tilde{\eta}(q) = -\beta'(L)$. Let $D \subset S_{r,U}$ be a small open neighborhood of q. There exists a small $\epsilon > 0$ such that the map

$$\Psi(t,y) := \tilde{\exp}_y(t\tilde{\eta}), \tag{3.26}$$

where $t \in [0,\epsilon]$ and $y \in D$, is a diffeomorphism from $[0,\epsilon] \times D$ onto its image in U. Define $D_\epsilon = \Psi(\epsilon, D)$. Clearly, $\hat{q} = \beta(L-\epsilon) \in D_\epsilon$. Moreover, when restricted to $[0, L-\epsilon]$, β minimizes distance between D_ϵ and S_U in (U, g). Using (3.19), by choosing ϵ sufficiently small, we further have

$$H(D_\epsilon, \hat{q}) > 0, \tag{3.27}$$

where $H(D_\epsilon, \hat{q})$ is the mean curvature of D_ϵ at $\hat{q}$ with respect to the outward normal in (M, g). Since $\hat{q} = \beta(L-\epsilon)$ is not a $\tilde{g}$-cut point to S_U along β, therefore repeating the previous argument with q replaced by $\hat{q}$ and $S_{r,U}$ replaced by D_ϵ, we again obtain a contradiction to the assumption that S_U is minimal in (M, g). This completes the proof. □

Remark 3.1. Among results from Section 2, we have only used Proposition 2.1 (i) to obtain the gradient estimate (3.2) in the proof of Theorem 3.1.

Remark 3.2. Theorem 3.1 does not depend on the dimension of M, i.e., it holds in all dimensions ≥ 3. Indeed, while the proof makes use of Proposition 2.1, it can be shown that this proposition extends to higher dimensions.

Remark 3.3. Even though in Theorem 3.1 we focus on asymptotically flat manifolds which are complete without boundary, it is clear from its proof that the conclusion of Theorem 3.1 holds for those (M, g) which have nonempty boundary ∂M (possibly disconnected) on which f vanishes.

4. Rigidity properties of static potentials

We now establish some rigidity properties of a noncompact, connected component (assuming such exists) of the zero set of a static potential f; cf. Theorem 4.2. As a corollary to this, we consider circumstances which imply global rigidity.

We continue to assume (M, g) is a complete, asymptotically flat 3-manifold without boundary, with finitely many ends. The triple (g, γ, ρ) is assumed to satisfy the decay assumptions (2.9) and (2.10) on each end.

The following is a fundamental (and almost immediate) consequence of Theorem 3.1.

Theorem 4.1. *Assume (M, g, γ, ρ) satisfies the NEC and f is a static potential. If Σ is a noncompact, connected component of $f^{-1}(0)$, then Σ is a strictly area minimizing surface in (M, g).*

Proof. By Lemma 2.1 (i), ∇f is a nowhere vanishing normal to Σ. Hence, there exists a connected open neighborhood W of Σ such that $W \setminus \Sigma$ is the disjoint union of two connected, unbounded, open sets W_+ and W_- satisfying $W_+ \subset \{f > 0\}$ and $W_- \subset \{f < 0\}$. Let U_+ and U_- be the connected component of $\{f > 0\}$ and $\{f < 0\}$ that contains W_+ and W_- respectively. As W_+, W_- are unbounded, so are U_+ and U_-.

To show that Σ is strictly area minimizing, we make use of solutions to the Plateau problem in (M, g) (cf. [1, 17]). Let $\{D_k\}_{k=1}^{\infty}$ be an exhaustion sequence of Σ such that each D_k is a bounded domain in Σ with smooth boundary Γ_k. (For instance, D_k may be taken as the subset of Σ enclosed by the union of large coordinates spheres near infinity on each end of (M, g) in which Σ extends to the infinity. By Prop. 2.2, each such sphere intersects Σ transversely.) Since (M, g) is asymptotically flat and hence foliated by mean convex spheres near infinity at each end, there exists a compact, embedded, minimal surface S_k in (M, g) such that $\partial S_k = \Gamma_k$ and S_k minimizes area among all compact surfaces having boundary Γ_k. By Theorem 3.1, $S_k \cap U_+ = \emptyset = S_k \cap U_-$. Therefore, $S_k \subset \Sigma$ and hence $S_k = D_k$. This shows that D_k strictly minimizes area among all surfaces with the same boundary. Since $\{D_k\}$ exhausts Σ, we conclude that Σ is strictly area minimizing. $\square$

So far we have only assumed the null energy condition. In what follows, we impose the stronger energy condition,

$$\rho \geq |\gamma(X, X)| \quad \text{for all unit vectors } X. \tag{4.1}$$

Note that if f is a static potential, then on the region $\{f \neq 0\}$ (whose closure is all of M) this energy inequality is a consequence of the spacetime dominant energy condition as applied to the static metric (1.1). Moreover, inequality (4.1) implies $R \geq 0$, where R is the scalar curvature of (M, g). Indeed, the trace of (1.5) and (1.6) imply that

$$\rho = \frac{1}{2} R \tag{4.2}$$

on the set $\{f \neq 0\}$, and hence, by continuity, on all of M.

Theorem 4.2. *Assume (M, g, γ, ρ) satisfies the energy condition* (4.1) *and f is a static potential. Suppose Σ is a noncompact, connected component of $f^{-1}(0)$. Then Σ is a strictly area minimizing, totally geodesic surface that is isometric to the Euclidean plane $\mathbb{R}^2$. Moreover, (M, g) is flat along Σ and $\rho = 0$, $\gamma = 0$ along Σ.*

Proof. Since (4.1) implies NEC, Σ is strictly area minimizing by Theorem 4.1. In particular, Σ is stable, i.e.

$$\int_\Sigma \left[|\nabla_\Sigma \eta|^2 - (\mathrm{Ric}(\nu,\nu) + |\mathbb{II}|^2)\eta^2\right] d\sigma \geq 0. \tag{4.3}$$

Here η is any Lipschitz function on Σ with compact support, ∇_Σ and $d\sigma$ denote the gradient and the area form on Σ respectively, ν is a unit normal along Σ, and $\mathbb{II}$ is the second fundamental form of Σ. By Lemma 2.1 (i), $\mathbb{II} = 0$. It follows from the Gauss equation that

$$2K = R - 2\mathrm{Ric}(\nu,\nu), \tag{4.4}$$

where K is the Gaussian curvature of Σ. Hence, (4.3) becomes

$$\int_\Sigma \left[|\nabla_\Sigma \eta|^2 - \left(\frac{1}{2}R - K\right)\eta^2\right] d\sigma \geq 0. \tag{4.5}$$

Now let $E_1, \ldots, E_k$ be those ends of (M,g) such that Σ extends to the infinity of E_i, $1 \leq i \leq k$. By Proposition 2.2, near the infinity of each E_i, $\Sigma = f^{-1}(0)$ which is the graph of some function $q = q(y_2, y_3)$ satisfying (2.11) in a coordinate chart $\{y_1, y_2, y_3\}$. For each large r, let $\gamma_r^{(i)} \subset \Sigma$ be the curve that is the graph of q over the circle $\{y_2^2 + y_3^2 = r^2\}$ in the y_2y_3-plane. Let $\Gamma_r = \cup_{i=1}^k \gamma_r^{(i)}$ and D_r be the bounded region in Σ enclosed by Γ_r. We claim

$$\mathrm{Area}(D_r) \leq C_1 r^2 \tag{4.6}$$

for some constant $C_1 > 0$. To see this, one can consider the map $F : \Omega_C \to \Sigma$ given by $F(y_2, y_3) = (q(y_2,y_3), y_2, y_3)$, where Ω_C is the exterior region in the y_2y_3-plane defined in Proposition 2.2. Let $\sigma = F^*(g)$ be the pulled back metric from Σ to Ω_C and let $\sigma_{\alpha\beta} = \sigma(\partial_{y_\alpha}, \partial_{y_\beta})$ where $\alpha, \beta \in \{2,3\}$. It follows from (2.9) and (2.11) that

$$\sigma_{\alpha\beta} = \delta_{\alpha\beta} + h_{\alpha\beta}, \tag{4.7}$$

where $h_{\alpha\beta}$ satisfies

$$|h_{\alpha\beta}| + |\bar{y}||\partial h_{\alpha\beta}| = O(|\bar{y}|^{-\tau}) \tag{4.8}$$

with $|\bar{y}| = \sqrt{y_2^2 + y_3^2}$. It is readily seen that (4.7) and (4.8) imply (4.6).

Next, we apply a logarithmic cut-off argument using (4.5) and (4.6). Given any large integer m, define η_m on Σ by

$$\eta_m(p) = \begin{cases} 1, & p \in D_{e^m} \\ 2 - \frac{\log|\bar{y}(p)|}{m}, & e^m \leq |\bar{y}(p)| \leq e^{2m} \\ 0 & p \notin D_{2e^m}. \end{cases} \tag{4.9}$$

Plugging $\eta = \eta_m$ in (4.5) gives

$$\begin{aligned} \int_\Sigma \left(\frac{1}{2}R - K\right)\eta_m^2 d\sigma &\leq \frac{1}{m^2}\int_{D_{e^{2m}}\setminus D_{e^m}} \frac{|\nabla_\Sigma|\bar{y}||^2}{|\bar{y}|^2}d\sigma \\ &\leq \frac{1}{m^2}\sum_{l=m+1}^{2m}\int_{D_{e^l}(p)\setminus D_{e^{l-1}}(p)} \frac{C_2}{|\bar{y}|^2}d\sigma \\ &\leq \frac{1}{m^2}C_1C_2\sum_{l=m+1}^{2m} e^{-2(l-1)}e^{2l} = \frac{1}{m}C_1C_2e^2, \end{aligned} \tag{4.10}$$

where we have used (4.6) and the fact $|\nabla_\Sigma|\bar{y}|| \leq |\nabla|\bar{y}|| \leq C_2$ for some constant $C_2 > 0$ when $\bar{y}$ is large. To proceed, we note that

$$R = O(|y|^{-\tau-2}) = O(|\bar{y}|^{-\tau-2}) \tag{4.11}$$

by (2.9), and

$$K = O(|y|^{-\tau-2}) = O(|\bar{y}|^{-\tau-2}) \tag{4.12}$$

by (2.9) and (4.4). These together with (4.7) and (4.8) imply

$$\int_\Sigma |R|d\sigma < \infty \quad \text{and} \quad \int_\Sigma |K|d\sigma < \infty. \tag{4.13}$$

Thus, letting $m \to \infty$ in (4.10), we have

$$\int_\Sigma K d\sigma \geq \frac{1}{2}\int_\Sigma R d\sigma. \tag{4.14}$$

On the other hand, by the Gauss-Bonnet theorem and (2.12) in Proposition 2.2,

$$\begin{aligned}\int_\Sigma K d\sigma &= \lim_{r\to\infty}\left(2\pi\chi(D_r) - \sum_{i=1}^{k}\int_{\gamma_r^{(i)}} \kappa\, ds\right)\\ &= 2\pi(\chi(\Sigma)-k)\\ &\le 0\end{aligned} \tag{4.15}$$

since $k \ge 1$ and the Euler characteristic $\chi(\cdot)$ of any connected, noncompact surface is at most 1. (So far we have not imposed the energy condition (4.1); thus (4.14) and (4.15) hold without a sign assumption on R.) Now applying the fact $R \ge 0$ (which is by (4.1) and (4.2)), we conclude from (4.14) and (4.15) that

$$\int_\Sigma K d\sigma = 0, \quad k = 1, \quad \chi(\Sigma) = 1, \quad \text{and} \quad R = 0 \text{ along } \Sigma. \tag{4.16}$$

Finally, we want to show $K = 0$ along Σ. We follow an argument of Fischer-Colbrie and Schoen in [12]. Note that $R = 0$ implies $K = -\mathrm{Ric}(\nu,\nu)$. Hence, the stability operator on Σ becomes $L = \Delta_\Sigma - K$, where Δ_Σ is the Laplacian on Σ. Since Σ is complete, by [12, Theorem 1], there exists a positive function v on Σ satisfying

$$\Delta_\Sigma v - Kv = 0. \tag{4.17}$$

Define $w = \log v$, then

$$\Delta_\Sigma w + |\nabla_\Sigma w|^2 = K. \tag{4.18}$$

Let η_m be given as in (4.9). Multiply (4.18) by η_m^2 and integrate by parts,

$$\begin{aligned}\int_\Sigma K\eta_m^2 d\sigma &= -2\int_\Sigma \eta_m\langle\nabla_\Sigma\eta_m, \nabla_\Sigma w\rangle d\sigma + \int_\Sigma \eta_m^2|\nabla_\Sigma w|^2 d\sigma\\ &\ge -\int_\Sigma\left(\frac{1}{2}\eta_m^2|\nabla_\Sigma w|^2 + 2|\nabla_\Sigma\eta_m|^2\right)d\sigma + \int_\Sigma \eta_m^2|\nabla_\Sigma w|^2 d\sigma\\ &= -2\int_\Sigma|\nabla_\Sigma\eta_m|^2 d\sigma + \frac{1}{2}\int_\Sigma\eta_m^2|\nabla_\Sigma w|^2 d\sigma.\end{aligned} \tag{4.19}$$

Let $m \to \infty$ and use the fact $\int_\Sigma |\nabla_\Sigma\eta_m|^2 d\sigma = O(m^{-1})$ (cf. (4.10)), we have

$$\int_\Sigma K d\sigma \ge \frac{1}{2}\int_\Sigma |\nabla_\Sigma w|^2 d\sigma. \tag{4.20}$$

Since $\int_\Sigma K d\sigma = 0$, this proves $\nabla_\Sigma w = 0$ on Σ, hence v is a positive constant. From (4.17), we conclude $K = 0$ along Σ. As a result, Σ is isometrically covered by $\mathbb{R}^2$. However, Σ cannot be a cylinder since $\chi(\Sigma) = 1$. Hence, Σ is isometric to $\mathbb{R}^2$.

To complete the proof, we note that $R = 0$ along Σ implies $\rho = 0$ and $\gamma = 0$ along Σ by (4.1) and (4.2). Therefore, by Lemma 2.1 (iii) and (2.3), (M, g) is flat along Σ. □

As an application of Theorem 4.2, we have

Corollary 4.1. *Assume (M, g, γ, ρ) satisfies the energy condition (4.1) and f is a static potential. Suppose that, outside a compact set, either $\rho > 0$ or (M, g) is non-flat at each point. Then f must be bounded on M and hence $f > 0$ or $f < 0$ near infinity on each end of (M, g).*

Proof. If f is unbounded on M, Proposition 2.2 implies that $f^{-1}(0)$ must have a noncompact, connected component Σ. By Theorem 4.2, (M, g) is flat and $(\rho, \gamma) = (0, 0)$ along Σ, which contradicts the assumption that (M, g) is non-flat or $\rho > 0$ outside a compact set. Therefore, f must be bounded on M. By Proposition 2.1 (ii), f is either positive or negative near infinity on each end of (M, g). □

Together with the positive mass theorem ([20, 21]), Corollary 4.1 implies the following rigidity result for asymptotically Schwarzschildean manifolds that admit an unbounded static potential.

Corollary 4.2. *Assume (M, g, γ, ρ) satisfies the energy condition (4.1) and f is a static potential. Suppose (M, g) is asymptotically Schwarzschildean at each end, i.e., there exists a coordinate chart $\{x_1, x_2, x_3\}$ near infinity on each end E_α, in which the metric g satisfies*

$$g_{ij} = \left(1 + \frac{m_\alpha}{2|x|}\right)^4 \delta_{ij} + p_{ij}, \tag{4.21}$$

where $|p_{ij}| + |x||\partial p_{ij}| + |x|^2|\partial^2 p_{ij}| = O(|x|^{-2})$ and m_α is some constant. Here $1 \le \alpha \le k$ and $E_1, \ldots, E_k$ denote all the ends of (M, g). Then, if f is unbounded, (M, g) is isometric to $\mathbb{R}^3$.

Proof. On each E_α, (4.21) implies that the Ricci curvature of g satisfies

$$\mathrm{Ric}(\partial_{x_i}, \partial_{x_j})(x) = \frac{m_\alpha}{|x|^3}\left[1 + \frac{m_\alpha}{2|x|}\right]^{-2}\left(\delta_{ij} - 3\frac{x_i x_j}{|x|^2}\right) + O(|x|^{-4}). \tag{4.22}$$

(See [14, Lemma 1.2] for instance). If $m_\alpha \neq 0$, $\forall\ \alpha = 1, \ldots, k$, then g is non-flat at each point outside a compact set in M. Thus f must be bounded on M by Corollary 4.1. This contradicts the assumption f is unbounded. Therefore, $m_\alpha = 0$ for some α.

On the other hand, by (4.21) and the fact $R \geq 0$, the ADM mass ([2]) of g at the end E_α exists and is equal to m_α. Hence it follows from the positive mass theorem that (M, g) is isometric to $\mathbb{R}^3$. □

Remark 4.1. Corollary 4.2 may be compared with [18, Theorem 1.1]. Corollary 4.2 is a global result for a complete, boundaryless, asymptotically Schwarzschildean manifold on which the static potential Equations (1.2) and (1.3) allow a nontrivial pair (ρ, γ). Theorem 1.1 in [18], proved for $(\rho, \gamma) = (0, 0)$, has a local feature that it is applicable on an asymptotically Schwarzschildean end.

Remark 4.2. We note that Corollary 4.2 also follows from Theorem 4.1 and a result of Carlotto in [6]. In [6, Theorem 1], Carlotto proved that if (M, g) is an asymptotically Schwarzschildean 3-manifold of nonnegative scalar curvature in which there exists a complete, noncompact, properly embedded stable minimal surface, then (M, g) is isometric to $\mathbb{R}^3$. The proof of [6, Theorem 1] also makes use of the positive mass theorem.

Work of Carlotto and Schoen [8] shows in a dramatic fashion that Carlotto's result no longer holds if the metric is merely assumed to obey the asymptotic condition (2.9) for $\tau \in (\frac{1}{2}, 1)$. However, Schoen [19] has raised the question as to whether there can be an area minimizing (not just stable) asymptotically planar minimal surface in a nontrivial asymptotically flat manifold (obeying this weaker decay) with nonnegative scalar curvature. In view of Theorem 4.1, these considerations lead us to believe that Corollary 4.2 remains valid under the weaker decay (2.9). In the vacuum case ($\rho = 0, \gamma = 0$) a spacetime approach which implies such rigidity follows from [4, 5] and references therein.

Final Remark. This paper was submitted for publication to CAG in December 2014, and accepted in October, 2015. Subsequently, a result has been obtained (cf. [7, Theorem 1.6]) which directly relates to the question of Schoen commented upon in Remark 4.2. It is shown in this result that the only asymptotically flat Riemannian three-manifold (having the general asymptotics (2.9)), with non-negative scalar curvature that admits a non-compact *area-minimizing boundary* is flat $\mathbb{R}^3$. As a consequence, [7, Theorem 1.6] can be combined with our Theorem 4.2 to conclude that Corollary 4.2 holds under the weaker asymptotics (2.9), as conjectured in Remark 4.2.

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